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Bound states of a charged particle and a dyon

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Abstract. Bound states of a charged particle in the field of a dyon are studied in the non-relativistic as well as the relativistic situations. In the relativistic case all angular momentum channels except the lowest are considered. Bound state energies and wavefunctions are calculated explicitly.

1. Introduction

In this paper, we shall study bound states of the system consisting of a charged particle and a dyon. By dyon we shall mean a Dirac monopole carrying electric charge. The charged particle will be coupled minimally to the electromagnetic field of the dyon. We shall first solve the non-relativistic problem. As is well known since the work of Dirac (1931), there is no non-relativistic bound state of the monopole–charge system. It is clear that if the monopole is given electric charge then bound states are possible for the resulting dyon–charge system, due to an attractive Coulomb force. In the next section, the bound state energies and wavefunctions are worked out explicitly. In § 3, we treat the problem in the relativistic single-particle approximation, i.e. treat the charged particle via the Dirac equation in the external field produced by the dyon. In the lowest angular momentum channel for this system, the eigenvalue problem is not mathematically well defined. This is due to the same reason as in the corresponding monopole–fermion problem (Kazama *et al* 1977). While a possible remedy for this has been suggested by Kazama *et al*, this requires going beyond the framework of the original problem by the introduction of an additional, non-minimal interaction due to the anomalous magnetic moment of the fermion. On the other hand, the limit of zero anomalous magnetic moment for the dyon–fermion problem is very problematic (Wu 1983). For all these reasons, we shall exclude consideration of the lowest angular momentum channel in this paper. In all other angular momentum channels the eigenvalue problem is well defined and the bound state energies and wavefunctions can be computed explicitly.

2. Non-relativistic bound states

We shall first treat the non-relativistic problem. Consider the Schrödinger equation

$$H\Psi = E\Psi \tag{1}$$

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where

$$H=\frac{1}{2M}(\boldsymbol{p}+e\boldsymbol{A})^2-\frac{eQ}{r} \tag{2}$$

and

$$\boldsymbol{A}=g\frac{1-\cos\theta}{r\sin\theta}\hat{\phi}. \tag{3}$$

In the above, $-e$ is the particle charge and Q is the charge of the dyon ($e>0$, $Q>0$). Also we use units such that $\hbar=c=1$. Equation (1) can be separated by setting

$$\Psi=\frac{u(r)}{r}S(\theta,\phi) \tag{4}$$

where $S(\theta, \phi)$ is the angular function of Tamm (1931). One may also avoid the use of the singular potential (3) by working in the two-patch formalism of Wu and Yang (1975) and using in place of $S(\theta, \phi)$ appropriate monopole harmonics (Wu and Yang 1976). The resulting radial equation obtained by these two procedures is, of course, one and the same. The desired radial equation, thus obtained, is

$$\frac{\mathrm{d}^2u}{\mathrm{d}r^2}+\left(2ME+\frac{2MeQ}{r}-\frac{J(J+1)-q^2}{r^2}\right)u=0 \tag{5}$$

where $q=eg$ and J is the angular momentum, i.e. $J(J+1)$ is the eigenvalue of the square of the conserved angular momentum operator $\boldsymbol{J}$, with

$$\begin{aligned}\boldsymbol{J}&=\boldsymbol{r}\wedge(\boldsymbol{p}+e\boldsymbol{A})+q\hat{r}\\&=\boldsymbol{r}\wedge\boldsymbol{p}-q\tan(\tfrac{1}{2}\theta)\hat{\theta}+q\hat{r}.\end{aligned} \tag{6}$$

As usual, a caret signifies a unit vector. We proceed to solve equation (5). First, we isolate the behaviour near the origin ($r\to0$), which is $u\sim r^a$, where

$$a=\tfrac{1}{2}+[(J+\tfrac{1}{2})^2-q^2]^{1/2}. \tag{7}$$

For bound states E must be negative and u exponentially damped at infinity. We define the variable $\rho=(-2ME)^{1/2}r$ and set

$$u=\rho^a\,\mathrm{e}^{-\rho}W(\rho). \tag{8}$$

Insertion of (8) into (5) gives

$$\tfrac{1}{2}\rho W''+(a-\rho)W'+(\tfrac{1}{2}\rho_0-a)W=0 \tag{9}$$

where $\rho_0=eQ(-2M/E)^{1/2}$ and a prime denotes differentiation with respect to ρ. Thus W is the confluent hypergeometric function:

$$W=A\,{}_1F_1(a-\tfrac{1}{2}\rho_0,2a,2\rho) \tag{10}$$

where A is a normalisation constant. The bound state enery levels follow directly from the above expression. In order not to upset the correct asymptotic behaviour of $u(\rho)$ at infinity, the function ${}_1F_1(a-\rho_0/2, 2a, 2\rho)$ must be a finite polynomial (of degree N, say). Thus

$$a-\tfrac{1}{2}\rho_0=-N \tag{11}$$

and therefore

$$E = \frac{-M(eQ)^2}{2(N+a)^2} \tag{12}$$

is the expression for the bound state energy levels. It is now instructive to study two limiting cases. In the limit $q=0$ ($g=0$), J is the orbital angular momentum l and $a=l+1$. The energy levels (12) and the wavefunction (10) now reduce to the well-known expressions for these quantities in the hydrogen atom problem (with $N+l+1$ being the principal quantum number). In the other limit set $Q=0$ to obtain the monopole-charge system. Now $\rho_0=0$ and equation (11) cannot be satisfied. We have recovered the well known result on the absence of non-relativistic bound states of the monopole-charge system. Moreover, taking note of the identity

$${}_1F_1(a, 2a, 2iz) = \Gamma(a+1/2)\left(\frac{z}{2}\right)^{1/2-a} e^{iz} J_{a-1/2}(z) \tag{13}$$

we see from (8) and (10) that the radial function u reduces to

$$u \sim (kr)^{1/2} J_{a-1/2}(kr) \tag{14}$$

which is the result of Tamm (1931). Here $k=(2mE)^{1/2}$ and $J_{a-1/2}$ is the Bessel function. Finally, the normalisation constant A in (10) may be determined. For u normalised as $\int u^2\,dr = 1$, this is given by

$$A = \frac{2^a (meQ)^{1/2}}{\sqrt{N!}(N+a)} \frac{[\Gamma(N+2a)]^{1/2}}{\Gamma(2a)}. \tag{15}$$

3. Relativistic bound states

Let us now turn our attention to the relativistic problem. Consider the Dirac equation

$$H\Psi = E\Psi \tag{16}$$

where

$$H = \boldsymbol{\alpha}\cdot(\boldsymbol{p}+e\boldsymbol{A}) + M\beta - eQ/r. \tag{17}$$

The conserved angular momentum operator corresponding to the above Hamiltonian is

$$\boldsymbol{J} = \boldsymbol{r}\wedge(\boldsymbol{p}+e\boldsymbol{A}) + q\hat{\boldsymbol{r}} + \tfrac{1}{2}\boldsymbol{\sigma}. \tag{18}$$

Let $J(J+1)$ denote the eigenvalue of the square of $\boldsymbol{J}$ and m that of J_z. The angular momentum analysis of equation (16) (for the case $Q=0$) has been given by Kazama *et al* 1977). Following this analysis we may conclude that, for our problem, the simultaneous eigenstates (or eigensections) of H, J^2 and J_z fall into three distinct types. Types 1 and 2 are characterised by the condition $J \geqslant |q|+\frac{1}{2}$ while type 3 by $J=|q|-\frac{1}{2}\geqslant 0$. As we can infer from the analysis of the monopole-fermion system, the radial equation corresponding to the type 3 solution $J=|q|-\frac{1}{2}$ does not lead to a mathematically well defined problem (Kazama *et al* 1977). For this reason we shall not discuss this case any further and confine our attention to the case $J\geqslant|q|+\frac{1}{2}$ throughout the remainder of this paper.

Setting, for type 1

$$\Psi^{(1)} = \frac{1}{r}\begin{pmatrix} F(r) & \xi^{(1)}_{Jm} \\ \mathrm{i}G(r) & \xi^{(2)}_{Jm} \end{pmatrix} \tag{19}$$

where $\xi^{(1)}_{Jm}$ and $\xi^{(2)}_{Jm}$ are the (two-component) angular functions given by Kazama *et al* 1977) and using the standard Dirac–Pauli representation of the matrices $\boldsymbol{\alpha}$ and β, we get the radial equations

$$\frac{\mathrm{d}F}{\mathrm{d}r} - \frac{\mu}{r}F = \left(M + E + \frac{eQ}{r}\right)G \tag{20a}$$

$$\frac{\mathrm{d}G}{\mathrm{d}r} + \frac{\mu}{r}G = \left(M - E - \frac{eQ}{r}\right)F. \tag{20b}$$

For type 2, we set

$$\Psi^{(2)} = \frac{1}{r}\begin{pmatrix} F(r) & \xi^{(2)}_{Jm} \\ \mathrm{i}G(r) & \xi^{(1)}_{Jm} \end{pmatrix} \tag{21}$$

to get the radial equations

$$\frac{\mathrm{d}F}{\mathrm{d}r} + \frac{\mu}{r}F = \left(M + E + \frac{eQ}{r}\right)G \tag{22a}$$

$$\frac{\mathrm{d}G}{\mathrm{d}r} - \frac{\mu}{r}G = \left(M - E - \frac{eQ}{r}\right)F. \tag{22b}$$

In the above

$$\mu = [(J + \tfrac{1}{2})^2 - q^2]^{1/2} \qquad \mu > 0. \tag{23}$$

Let us put

$$\alpha_1 = M + E \qquad \alpha_2 = M - E \qquad \gamma = eQ \qquad \rho = r(\alpha_1\alpha_2)^{1/2} \tag{24}$$

to rewrite equation (20) in the form

$$F' - \frac{\mu}{\rho}F = \left((\alpha_1/\alpha_2)^{1/2} + \frac{\gamma}{\rho}\right)G \tag{25a}$$

$$G' + \frac{\mu}{\rho}G = \left((\alpha_2/\alpha_1)^{1/2} - \frac{\gamma}{\rho}\right)F. \tag{25b}$$

The corresponding equations for type 2 follow from the above by replacing μ with $-\mu$. We proceed to solve these radial equations. We first isolate the behaviour near the origin $\rho \to 0$, which is

$$F \sim \rho^s \qquad G \sim \rho^s \tag{26}$$

where

$$s = (\mu^2 - \gamma^2)^{1/2} \tag{27}$$

for both type 1 and type 2. We thus seek bound state solutions of the form

$$\begin{aligned} F &= \alpha_1^{1/2}\rho^s\,\mathrm{e}^{-\rho}(\psi_1 + \psi_2) \\ G &= \alpha_2^{1/2}\rho^s\,\mathrm{e}^{-\rho}(\psi_1 - \psi_2). \end{aligned} \tag{28}$$

From (25*a*), (25*b*) and (28) we get the decoupled equations

$$\psi_1'' + \left(\frac{2s+1}{\rho} - 2\right)\psi_1' - \frac{2}{\rho}\left(1 + s - \frac{\gamma\varepsilon}{(1-\varepsilon^2)^{1/2}}\right)\psi_1 = 0 \tag{29}$$

$$\psi_2'' + \left(\frac{2s+1}{\rho} - 2\right)\psi_2' - \frac{2}{\rho}\left(s - \frac{\gamma\varepsilon}{(1-\varepsilon^2)^{1/2}}\right)\psi_2 = 0 \tag{30}$$

where a prime denotes a derivative with respect to ρ. It follows, from the above, that (up to multiplicative constants)

$$\psi_1 = {}_1F_1(1-n', 2s+1, 2\rho) \qquad \psi_2 = {}_1F_1(-n', 2s+1, 2\rho) \tag{31}$$

where

$$n' = \frac{\gamma\varepsilon}{(1-\varepsilon^2)^{1/2}} - s \qquad \varepsilon = \frac{E}{M} \tag{32}$$

and ${}_1F_1$ is the confluent hypergeometric function. Equations (29) and (30) and the solution (31) remain valid for type 2 solution, as well. Collecting the above results, we obtain finally

$$F = A\alpha_1^{1/2}\rho^s\, e^{-\rho}[{}_1F_1(1-n', 2s+1, 2\rho) + c\, {}_1F_1(-n', 2s+1, 2\rho)] \tag{33}$$

$$G = A\alpha_2^{1/2}\rho^s\, e^{-\rho}[{}_1F_1(1-n', 2s+1, 2\rho) - c\, {}_1F_1(-n', 2s+a, 2\rho)] \tag{34}$$

where A is an overall normalisation constant. The relative normalisation constant c is determined from the first-order equations such as (25*a*) or (25*b*) and, therefore, does depend on the solution type. We find

$$c = \begin{cases} -\dfrac{1}{n'}\left(\mu + \dfrac{\gamma}{(1-\varepsilon^2)^{1/2}}\right) & \text{for type 1} \\ \dfrac{1}{n'}\left(\mu - \dfrac{\gamma}{(1-\varepsilon^2)^{1/2}}\right) & \text{for type 2.} \end{cases} \tag{35}$$

Finally, to maintain the correct behaviour as $\rho \to \infty$, the functions $\rho^{-s}\, e^{\rho}F$ and $\rho^{-s}\, e^{\rho}G$ must be finite polynomials (of degree N, say). We thus obtain

$$n' = N \tag{36}$$

with N a non-negative integer. Equations (32) and (36) give the desired energy levels

$$E = \frac{M}{[1 + \gamma^2/(s+N)^2]^{1/2}}. \tag{37}$$

Thus type 1 and type 2 solutions have the same energy. This generalises the well known two-fold degeneracy of the Coulomb problem. Again, one may verify that the above expressions possess the correct behaviour in the two limiting cases. In the limit $q=0$, equations (33)-(37) reduce to well known expressions for the Coulomb problem. In the other limit $\gamma = 0$, condition (36) cannot be satisfied and there are no bound states of the monopole-fermion system. This confirms the conclusion of Harish-Chandra (1948). Furthermore, in this limit $c=+1$ for type 1 and $c=-1$ for type 2 and one finds straightforwardly that

$$\begin{aligned} &F \sim \rho^{1/2} J_{\mu-1/2}(kr) \qquad G \sim \rho^{1/2} J_{\mu+1/2}(kr) \qquad \text{for type 1} \\ &F \sim \rho^{1/2} J_{\mu+1/2}(kr) \qquad G \sim \rho^{1/2} J_{\mu-1/2}(kr) \qquad \text{for type 2} \end{aligned} \tag{38}$$

which is the correct behaviour for the monopole-fermion system (Kazama *et al* 1977). Finally, the overall normalisation constant A in (33) may be determined. For Ψ normalised to unity, this constant is easily determined to be

$$A = 2^s \frac{N}{\sqrt{N!}} \frac{[\Gamma(N+2s+1)]^{1/2}}{\Gamma(2s+1)} \left[\frac{(1-\varepsilon^2)^{1/2}}{N^2c^2+N(N+2s)}\right]^{1/2}$$

$$\equiv 2^s \frac{N}{\sqrt{N!}} \frac{[\Gamma(N+2s+1)]^{1/2}}{\Gamma(2s+1)} \left(\frac{1-\varepsilon^2}{2\gamma}\right)^{1/2} \left(\frac{1}{-cN}\right)^{1/2}. \tag{39}$$

It is easy to check from the above expressions (33)-(35) and (39) that the type 2 solution vanishes identically for the special case $N=0$. This is as it should be.

4. Concluding remarks

We have solved two problems. First, the bound states of a non-relativistic spinless particle and a dyon and, second, those of a relativistic spin-$\frac{1}{2}$ particle and a dyon. It should be emphasised that these two problems are really quite different. In particular, it is not meaningful to compare the non-relativistic limit of the Dirac theory with the Schrödinger theory. This is because the spectrum of angular momentum is different for the two cases plus the fact that the energy levels do depend on angular momentum, even in the Schrödinger approximation. However, one can easily write down a Schrödinger-Pauli theory and this provides the desired non-relativistic limit of the Dirac theory.

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